Formulation Method for Solid-to-Beam Transition Finite Elements

Jang-Keun Lim*
Professor, School of Mechanical Engineering, Hanyang University, 17 Haengdang-Dong, Sungdong-Ku, Seoul, 133-791, Korea
Tae-Han Song, Byung-Ho Seog
Research Assistant, School of Mechanical Engineering, Hanyang University, Seoul 133-791, Korea

Various transition elements are used in general for the effective finite element analysis of complicated mechanical structures. In this paper, a solid-to-beam transition finite element, which can be used for connecting a C^1-continuity beam element to a continuum solid element, is proposed. The shape functions of the transition finite element are derived to meet the compatibility condition, and a transition element equation is formulated by the conventional finite element procedure. In order to show the effectiveness and convergence characteristics of the proposed transition element, numerical tests are performed for various examples. As a result of this study, following conclusions are obtained. (1) The proposed transition element, which meets the compatibility of the primary variables, exhibits excellent accuracy. (2) In case of using the proposed transition element, the number of nodes in the finite element model may be considerably reduced and the model construction becomes more convenient. (3) This formulation method can be applied to the usage of higher order elements.

Key Words : Transition Finite Element, Hexahedral Solid Element, Quadrilateral Plane Element, Euler's Beam Element, Isoparametric Formulation Method, Compatible Displacement Field.

1. Introduction

In analyzing a structure by the finite element method, the selection of finite elements is very important for the solution accuracy and reliability. Especially, it is difficult to model a complicated mechanical structure using structural elements only, so a continuum element is used in general (Bathe, 1982; Reddy, 1993). However, when complicated mechanical structures are modeled using the continuum elements, considerable efforts and time are required for preparing the input data. For a plane problem, these input data may be prepared without much difficulty, but for 3D solid problems their preparation is not easy. Thus, nowadays, the element construction and the preparation of input data are automatically executed by the mesh generation module of commercial package programs. Users who do not know well the characteristics of the generation module and the generated elements, however, may not obtain good results. So, in order to efficiently analyze a complex structure and reduce the efforts and time required for the analysis, several transition elements are needed in general.

These days, the transition finite elements are classified into two types. The first type is used as an element for the transition region where the element discretization changes from higher order to lower order or from coarse mesh to fine mesh.

* Corresponding Author,
E-mail : jklim@email.hanyang.ac.kr
TEL : +82-2-2290-0441; **FAX** : +82-2-2281-1869
Professor, School of Mechanical Engineering, Hanyang University, 17 Haengdang-Dong, Sungdong-Ku, Seoul, 133-791, Korea. (Manuscript **Received** March 24, 2001; **Revised** August 23, 2001)

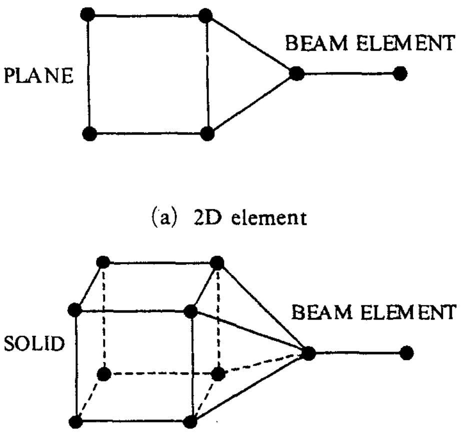
This transition element has been studied for increasing continuity and compatibility (Cook et al., 1989; Gupta, 1978), and is mainly used for the adaptive finite element analysis. The other type is used to connect elements that are different in kind. Because the degrees of freedom and the primary variables of nodes between the elements being connected are different, the kinematic compatibility of this transition element has to be seriously considered.

For the transition element of the latter type, a solid-to-beam element proposed by T. C. Gmür and R. H. Kauten (1993) is used for the connection of solid elements with the Timoshenko's beam elements. But this transition element, formulated by the isoparametric formulation method (Gmür and Kauten, 1993; Surana, 1980), is mathematically complex, and can not be used with the Euler's beam element due to the lack of compatibility.

In this paper, a formulation method of a solid-to-beam transition finite element that can be used for the connection of a continuum element with an Euler beam element is newly proposed. Low order 2D plane elements and 3D solid elements are used as the continuum element, and the beam element considered is the C^1-continuity Euler beam element. Shape functions at nodes where the continuum and beam elements are connected are determined to meet the compatibility of the primary variables. Using these shape functions, a matrix equation for the transition element is formulated by the conventional finite element procedure. Through various numerical tests, the accuracy and convergence characteristics of the proposed element are studied.

2. Formulation Method

A continuum element and a beam element are different in kind. Various transition elements that connect these elements can be considered in accordance with the orders and kinds of finite elements used in the analysis. In this paper, as shown in Fig. 1, a solid-to-beam transition element through which a low order continuum



(a) 2D element

(b) 3D element

Fig. 1 Solid-to-beam transition elements

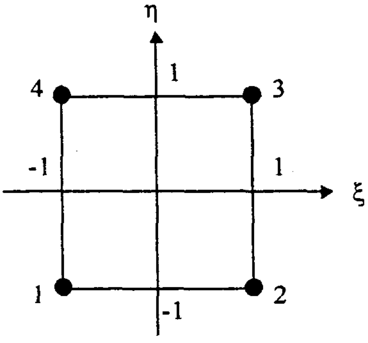


Fig. 2 4-noded quadrilateral plane element

element and an Euler beam element can be connected is considered. Figure 1(a) shows a 2-D transition element that connects a 4-noded quadrilateral plane element with a plane beam element, and Fig. 1(b) shows a 3D transition element that connects a 8-noded hexahedral solid element with a general beam element.

In this work, the plane and solid elements are referred to as continuum elements, and are both formulated using the isoparametric method. The mapped geometry of the 4-noded quadrilateral plane element is depicted in Fig. 2, and its mapping or shape functions are given as follows (Bathe, 1982; Cook et al., 1989; Gupta, 1978; Reddy, 1993).

$$N_1=\frac{1}{4}(1-\xi)(1-\eta),\ N_2=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3=\frac{1}{4}(1+\xi)(1+\eta),\ N_4=\frac{1}{4}(1-\xi)(1+\eta) \quad (1)$$

On the other hand, the 8-noded hexahedral solid element is isoparametrically mapped as shown in Fig. 3, and its mapping or shape functions are expressed in Eq. (2).

$$\begin{aligned}
N_1&=\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta),\ N_2=\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)\\
N_3&=\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta),\ N_4=\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)\\
N_5&=\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta),\ N_6=\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)\\
N_7&=\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta),\ N_8=\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta) \quad (2)
\end{aligned}$$

In Eqs. (1) and (2), ξ, η and ζ are natural coordinates which are between -1 and $+1$. The primary variables of these continuum elements, having the C^0-continuity, are the displacements in the x, y and z directions. The Euler's beam element, which is connected with them, however has C^1-continuity, and thus the primary variables must be taken as displacements and their 1st-order derivatives. Figure 4 shows the nodal degrees of freedom of a 3D Euler beam element.

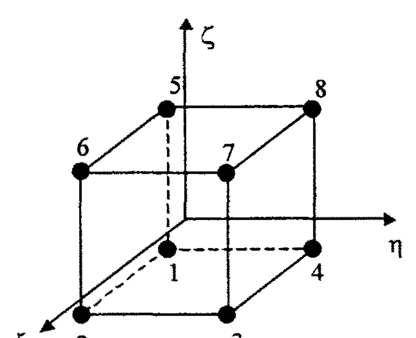


Fig. 3 8-noded hexahedral solid element

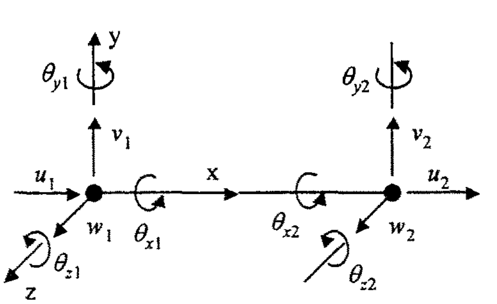


Fig. 4 Nodal degrees of freedom of a 3D Euler beam element

In the Euler beam theory, the displacements in each direction, u_x, u_y, u_z, and the rotational angle, θ_x, θ_y, θ_z, hold the following relations (Cook et al., 1989; Reddy, 1993).

$$\begin{aligned}
u_x&=u+z\theta_y-y\theta_z\\
u_y&=v-z\theta_x\\
u_z&=w+y\theta_x \quad (3)
\end{aligned}$$

In Eq. (3), u, v and w are the neutral axis displacements in the x, y and z directions, respectively; and, θ_x is the torsional angle and θ_y, θ_z denote the deflection angles in the y and z directions, respectively, and have the following relations with the displacements.

$$\begin{aligned}
\theta_y&=-\frac{dw}{dx}\\
\theta_z&=\frac{dv}{dx} \quad (4)
\end{aligned}$$

Now, for the Euler beam element shown in Fig. 4, the displacement shape functions satisfying Eqs. (3) and (4) are found in the following form.

$$\{u\}=[N]\{d\} \quad (5)$$

In Eq. (5), $\{u\}$ denotes the displacement vector defined by $\{u\}^T=[u_x\ u_y\ u_z]$, and $\{d\}$ is the nodal displacement vector of an element defined by $\{d\}^T=[u_1\ v_1\ w_1\ \theta_{x1}\ \theta_{y1}\ \theta_{z1}\ u_2\ v_2\ w_2\ \theta_{x2}\ \theta_{y2}\ \theta_{z2}]$. On the other hand, the shape function matrix $[N]$ is expressed as

$$[N]=\begin{bmatrix} N_1 & -\frac{a}{2}\eta N_3' & -\frac{b}{2}\zeta N_3' & 0 & \frac{b}{2}\zeta N_4' & -\frac{a}{2}\eta N_4' & N_2 & -\frac{a}{2}\eta N_5' & -\frac{b}{5}\zeta N_5' & 0 & \frac{b}{2}\zeta N_6' & -\frac{a}{2}\eta N_6' \\ 0 & N_3 & 0 & -\frac{b}{2}\zeta N_1 & 0 & N_4 & 0 & N_5 & 0 & -\frac{b}{2}\zeta N_5' & 0 & N_6 \\ 0 & 0 & N_3 & \frac{a}{2}\eta N_1 & -N_4 & 0 & 0 & 0 & N_5 & \frac{a}{2}\eta N_2 & -N_6 & 0 \end{bmatrix} \quad (6)$$

where

$$\begin{aligned}
&N_1=1-\xi,\ N_2=\xi\\
&N_3=1-3\xi^2+2\xi^3,\ N_4=l\xi(1-\xi)^2\\
&N_5=3\xi^2-2\xi^3,\ N_6=l(-\xi^2+\xi^3) \quad (7)
\end{aligned}$$

In the equations above, the non-dimensional coordinates are re-defined as $\xi=\frac{x}{l}$, $\eta=\frac{2y}{a}$ and $\zeta=\frac{2z}{b}$ which are between -1 and $+1$; l is the length; and, a and b represent the height and width in the y and z directions, respectively, of the beam element. And, N_i' denotes the 1st-order derivative of N_i with respect to x.

2.1 Plane transition element

Now, in order to find the shape functions of the transition element that connects the continuum element with the beam element, a 2D three-noded transition element shown in Fig. 5 considered.

In Fig. 5, nodes 1 and 2 are the nodal points being connected with a continuum element, and node 3 is the nodal point to which an Euler beam element having the C^1-continuity is linked. Therefore, the shape functions corresponding to these nodes can be obtained from Eqs. (1) and (6), and using these shape functions, the displacement components of the transition element are expressed as

$$\{u\}_2=[N]_2\{d\}_2, \tag{8}$$

where

$\{d\}_2^T=[u\ v]$ and $\{d\}_2^T=[u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ \theta_{z3}]$.

In Eq. (8), the shape function matrix $[N]_2$ is obtained as

$$[N]_2=\begin{bmatrix} N_1^s & 0 & N_2^s & 0 & N_1^b & 0 & -\frac{a}{2}\eta N_1^b \\ 0 & N_1^s & 0 & N_2^s & 0 & N_2^b & N_3^b \end{bmatrix} \tag{9}$$

where

$$N_1^s=\frac{1}{2}(1-\xi)(1-\eta),\ N_2^s=\frac{1}{2}(1-\xi)(1+\eta)$$
$$N_1^b=\xi,\ N_2^b=3\xi^2-2\xi^3,\ N_3^s=l(\xi^2-\xi^3) \tag{10}$$

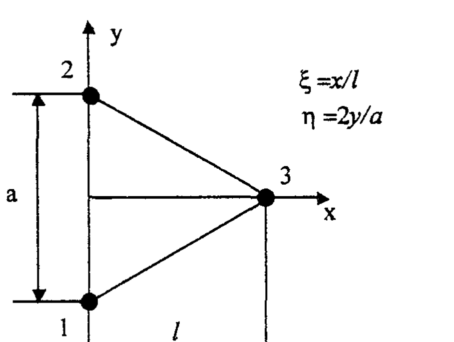


Fig. 5 Two-dimensional transition element with three nodes

2.2 Solid transition element

As shown in Fig. 6, another transition element that connects a 3D solid element with a general beam element is assumed to be a simple 5-noded quadrangular pyramid.

In Fig. 6, nodes 1, 2, 3 and 4 are the nodal points connected with a hexahedral solid element, and node 5 is the nodal point to which a general Euler beam element having the C^1-continuity is linked. Therefore, the shape functions corresponding to these nodes can be found from Eqs. (1) and (6), and using these shape functions, the displacement components of the transition element are expressed as

$$\{u\}_3=[N]_3\{d\}_3 \tag{11}$$

where $\{u\}_3^T=[u\ v\ w]$ and $\{d\}_3^T=[u_1\ v_1\ w_1\ u_2\ v_2\ w_2\ u_3\ v_3\ u_4\ v_3\ w_4\ u_5\ v_5\ w_5\ \theta_{x5}\ \theta_{y5}\ \theta_{z5}]$.

In Eq. (11), the shape function matrix $[N]_3$ is obtained as

$$[N]_3=[N]_s+[N]_b \tag{12}$$

$$[N]_s=\begin{bmatrix} N_1^s & 0 & 0 & N_2^s & 0 & 0 & N_3^s & 0 & 0 & N_4^s & 0 & 0 \\ 0 & N_1^s & 0 & 0 & N_2^s & 0 & 0 & N_3^s & 0 & 0 & N_4^s & 0 \\ 0 & 0 & N_1^s & 0 & 0 & N_2^s & 0 & 0 & N_3^s & 0 & 0 & N_4^s \end{bmatrix} \tag{13a}$$

$$[N]_b=\begin{bmatrix} N_1^b & 0 & 0 & 0 & \frac{b}{2}\zeta N_1^b & -\frac{a}{2}\eta N_1^b \\ 0 & N_2^b & 0 & -\frac{b}{2}\zeta N_1^b & 0 & -N_3^b \\ 0 & 0 & N_2^b & \frac{a}{2}\eta N_1^b & N_3^b & 0 \end{bmatrix} \tag{13b}$$

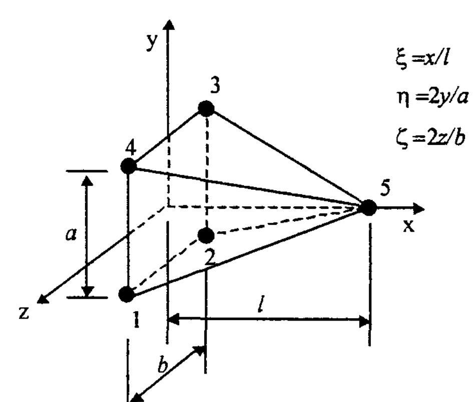
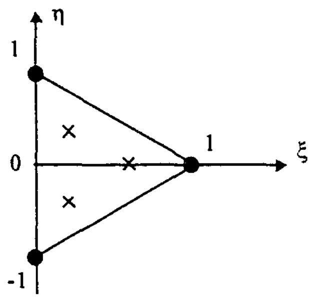
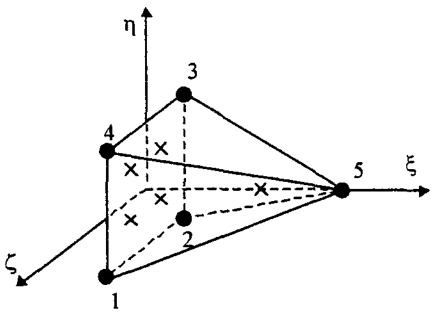


Fig. 6 Three-dimensional transition element with five nodes



(a) Plane



(b) Solid

Fig. 7 Gauss quadrature schemes for the transition elements

where

$$N_1^s=\frac{1}{4}(1-\xi)(1-\eta)(1-\zeta)$$
$$N_2^s=\frac{1}{4}(1-\xi)(1+\eta)(1-\zeta)$$
$$N_3^s=\frac{1}{4}(1-\xi)(1+\eta)(1+\zeta)$$
$$N_4^s=\frac{1}{4}(1-\xi)(1-\eta)(1+\zeta) \tag{14}$$

And, the shape functions, N_1^b, N_2^b and N_3^b, are given by the expressions in Eq. (10). Now, using the shape functions expressed in Eqs. (9) and (12), the matrix equations of the transition finite elements are found through the same procedure as in the conventional finite element method (Bathe, 1982; Cook et al., 1989; Reddy, 1993).

2.3 Numerical integration

The stiffness matrix of the transition element obtained from the equations above is not mathematically complex and so its integration can be done in a closed form. The stiffness matrix obtained by the closed form integration, however, may be overestimated in certain cases (Cook et al., 1989). In order to properly evaluate the stiffness matrix, numerical integration is performed. Figure 7 shows the Gauss quadrature scheme for the transition elements. In Fig. 7, x denotes the Gauss integration points, and both the plane and solid transition elements are integrated by two point Gauss quadrature (Cook. et al., 1989) but by one point near an apex.

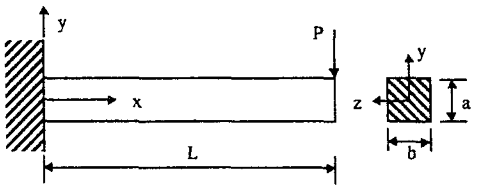


Fig. 8 Cantilever with rectangular section subjected to a tip load

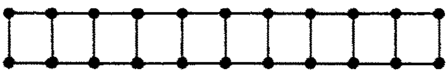
3. Numerical Examples

In order to examine the accuracy and convergence characteristics of the proposed transition elements, numerical examples are analyzed using several finite elements and the results are compared with each other.

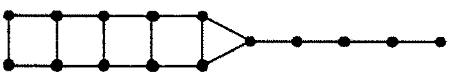
3.1 Cantilever

Firstly, as shown in Fig. 8, a cantilever with rectangular section (a×b) subjected to a concentrated end load P are analyzed using several finite elements. Figure 9 shows several finite element models of the cantilever: (a) a model consisting of 10 plane elements only, (b) a mixed model with a ransition element, and (c) a model consisting of 10 beam elements only. The mixed model is effective in that the number of nodes is reduced by 22 % compared to the model of plane elements only.

The variation of nodal displacements along the beam length is compared in Fig. 10. It is shown that the analysis result of the mixed model with a transition element is closer to the presumed exact solution, obtained using beam elements only, than



(a) Plane element



(b) Transition element

(c) Beam element

Fig. 9 Two-dimensional finite element models of the cantilever

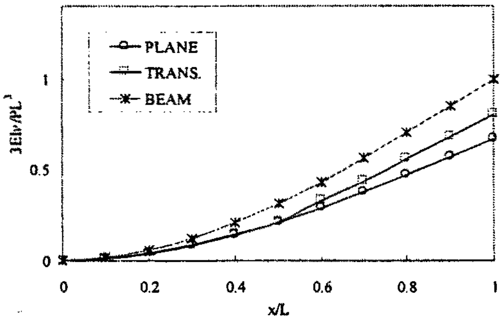
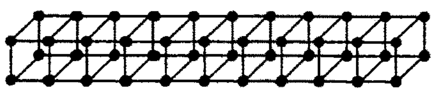
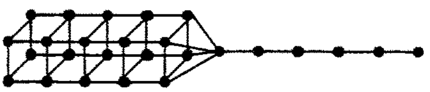


Fig. 10 Nodal displacement variation along the beam length for various finite element modeling methods



(a) Solid element



(b) Transition element

(c) Beam element

Fig. 11 Three-dimensional finite element models of the cantilever

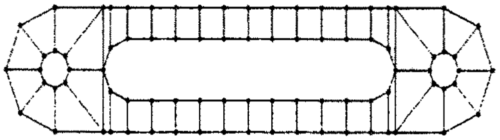
that of the model of plane elements only.

Figure 11 shows the finite element model for three-dimensional analysis of the cantilever shown in Fig. 8. Figure 11(a) shows a model of 10 solid elements only, (b) a mixed model with a transition element, and (c) a model of 10 beam elements only.

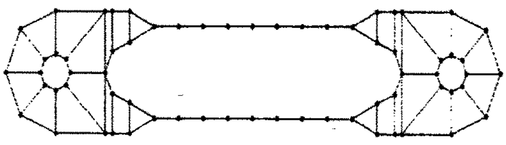
The analysis results are similar to those shown in Fig. 10, and the result of the mixed model with a transition element is more accurate than that of the model of solid elements only. Thus, we can state that the use of a transition element increases the solution accuracy and convergence, and makes the finite element modeling be more convenient.

3.2 Connecting rod

A connecting rod is a typical structure for which the plane transition elements can be useful in finite element modeling. As shown in Fig. 12, the parts of a connecting rod can be modeled using plane elements or beam elements. Figure 12 (a) shows a model of plane elements only, and (b) a mixed model with 4 plane transition elements. The analysis results show the same characteristics as those for the cantilever.

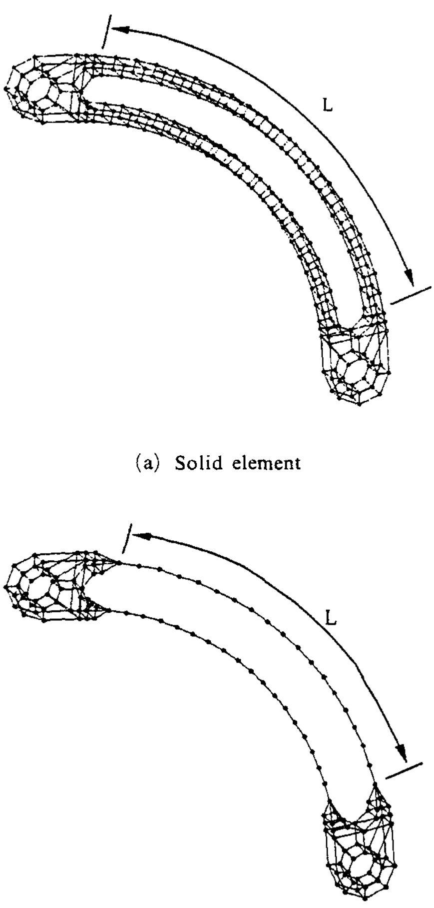


(a) Plane element



(b) Transition element

Fig. 12 Two-dimensional finite element models of connecting rod



(a) Solid element

(b) Transition element

Fig. 13 Three-dimensional finite element models of curved link

3.3 Curved link

A curved link can be regarded as a typical structure for which the solid transition elements may be useful in finite element modeling. As shown in Fig. 13, the parts of a long link can be modeled using solid elements or beam elements. Figure 13(a) shows a model of solid elements only, and Fig. 13(b) a mixed model with 4 solid transition elements. The analysis results also show the same characteristics as the previous results.

Therefore, the use of the proposed transition elements makes the finite element modeling of a complex structure be relatively convenient, and increases the solution accuracy and convergence characteristics in virtue of the compatibility of the primary variables.

4. Conclusion

In order to analyze effectively a complicated mechanical structure by the finite element method, transition elements are formulated. A 3-noded plane transition element and a 5-noded solid transition element are newly proposed to connect a continuum element and an Euler's beam element. Through various numerical examples, it is shown that these proposed transition elements enhance the efficiency of the finite element analysis. As a result of this study, following conclusions are obtained.

(1) The proposed transition elements, which meet the compatibility of the primary variables, yield good accuracy.

(2) If these transition elements are used, the number of nodes in a finite element model may be considerably reduced, and the model construction may become more convenient.

(3) The formulation method of these transition elements can be applied for higher order elements.

References

Bathe, K. J., 1982, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, N. J.

Cook, R. D., Malkus, D. S. and Plesha, M. E., 1989, *Concept and Applications of Finite Element Analysis, 3rd Ed.*, John Wiley & Sons, New York.

Gmür, T. C. and Kauten, R. H., 1993, "Three-Dimensional Solid-to-Beam Transition Elements for Structural Dynamics Analysis," *International Journal for Numerical Methods in Engineering*, Vol, 36, pp. 1429~1444.

Gupta, A. K., 1978, "A Finite Element for Transition from a Fine to a Coarse Grid," *International Journal for Numerical Methods in Engineering*, Vol, 12, No. 1, pp. 35~45.

Reddy, J. N., 1993, *An Introduction to the Finite Element Method, 2nd Ed.*, McGraw-Hill, New York

Surana, K. S., 1980, "Shape Functions for the Isoparametric Transition Elements for Cross-Sectional Properties and Stress Analysis of Beams," *International Journal for Numerical Methods in Engineering*, Vol, 15, pp. 1403~1407.